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The High Road/Low Road Demonstration or Birds on a Wire

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Abstract

Consider two separate tracks of equal horizontal displacements and equal initial and final heights. One track remains at this initial height while the other angles down, levels out, and then angles back up in order to regain its original height. Question: If two identical balls are set rolling with equal initial speeds, which ball completes the track in a shorter time interval? In this manuscript, the dynamics of a ball on each track are analyzed using basic Newtonian mechanics. We calculate the time necessary to complete each path in terms of the parameters of the track and the initial velocities of the balls. We derive an expression for the time difference between the two tracks and compare this to data taken on a set of high road/load road tracks, hence demonstrating the fact that the ball traversing the low road *always* wins the race.

Key words: conservation of energy, kinematics, moment of inertia, Newtonian mechanics, rolling without slipping

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I. INTRODUCTION

Consider two separate tracks of equal horizontal distance and initial and final heights. The flat track remains at this constant initial height while the second track dips down into a flat valley, hence following a longer path, before regaining its original height. An example of such a set of tracks is displayed in Fig. 1. Question: If two identical balls are set rolling with equal initial speeds, which ball completes the track in a shorter time interval? The ball that takes the deviated path into the valley will of course travel at a higher speed for much of the race, however, that ball has a longer distance to travel. On first glance, the answer may not seem at all obvious. One of us first encountered the aforementioned question/ demonstration at the *13th Workshop for New Physics and Astronomy Faculty* where the workshop attendees failed to reach a consensus on the correct outcome prior to witnessing the demonstration. In fact, Leonard et. al. found that about two-thirds of introductory physics students wrongly predict that both balls will complete the track in equal times. One reason often cited for this prediction involves invoking conservation of energy to (correctly) identify that the ball will have the same initial and final speeds and conclude (incorrectly) that the balls must therefore tie. Further, about one-sixth of intro students incorrectly predict that the ball following the flat track will reach the other end first (as they travel the shorter distance). Only about one-tenth correctly identify that the ball following the longer path will in fact win the race [1]. A further study of students' judgments concerning the outcomes of races involving such a pair of tracks is presented in [2].

The answer to the above conceptual question becomes obvious when the velocity components are analyzed. When the dissipative effects of friction are ignored, the speed of the ball traversing the flat track remains constant. For the ball that follows the longer path, the horizontal component of the velocity is always equal to or larger than that of the speed of the ball on the flat track. This is due to the fact that the ball experiences a horizontal net force on the first diagonal section of the track causing the ball to undergo a horizontal acceleration. Although the ball does in fact experience a horizontal deceleration on second diagonal section, the ball's horizontal component of velocity will always remain at or above the initial speed of the ball. *Thus, the ball traversing the longer path always wins the race, regardless of the initial speed.* This outcome is also independent of the breadth of the flat valley with a similar result emerging from a V-valley track [1, 2].

As is well known, Johann Bernoulli (1667-1748) found the solution of the path of shortest travel time between two points for an object under the influence of a constant gravitational force. This pioneering work began the study of the calculus of variations. In fact, the brachistochrone is often first introduced in an advanced undergraduate dynamics class as an example of an application of the calculus of variations. Since this original work, there have been several follow up analyses of generalizations of the brachistochrone and tautochrone including [3–11]. Several articles have been written highlighting the incorporation of the brachistochrone into the classroom, see for example [12–14].

Zheng et. al. considered a race involving beads of equal initial speeds on a straight and curvilinear path [15] and found that the outcome of the race was dependent on the initial speed (see [14] for a similar result involving spherical balls on a straight path vs. the brachistochrone curve). This result differs from that of the high road/low road where the ball on the low road always wins, independent of the initial speed. This apparent inconsistency arises from the fact that the scenario of Zheng et. al. is fundamentally different than that of the high road/low road and can be understood in terms of the orientation of the initial velocity vectors. For two non-parallel initial velocity vectors of equal magnitudes, the horizontal components differ and thus allow for a race dependent on the initial speed.

In section II, we calculate the time interval for a spherical ball to traverse each track in terms of the parameters of the track and the initial speed of the ball. From this, one can easily construct the time difference. The derivation of the time interval of the low road consists of dissecting the track into five distinct regions and calculating the time needed to traverse each. In section III, we discuss the experimental design and setup of our high road/low road apparatus. In section IV, we discuss our experimental procedure and present the results as a plot of the time difference versus the average speed of the ball.

FIG. 1: An example of a high road/low road track

II. THEORETICAL ANALYSIS

The two tracks studied here are equal in horizontal displacement and are symmetrical about their midpoints. Both tracks begin and end at the same initial height ($h_{high,i} = h_{low,i} = h_{high,f} = h_{low,f}$). The tracks differ in that one dips down in its midsection while the height of the other remains constant. Throughout this paper, *low road* will refer to the track that does in fact deviate in the midsection; *high road* will refer to the track that remains at constant height. A diagram of the first half of the low road is illustrated as the red line in Fig. 2, which includes other relevant track parameters.

Each symmetric half of the low road is partitioned into five sections. The upper horizontal is the first flat reach that is level with the high road. This track then enters an area of constant radius of curvature, where the track deviates from the upper horizontal, before beginning the straight diagonal section. The track then enters another area of constant radius of curvature before reaching the second flat reach of the lower horizontal section. The radii of curvature of both arcs are identical and were conveniently designed and constructed to be equal to the elevation difference between the higher and lower horizontal sections of the low road.

A. The Horizontal Sections of the High and Low Road

We begin by first calculating the time needed to traverse the high road. As the ball undergoes zero acceleration, the time is simply given by

$$t_H = \frac{L}{v_1}, \tag{2.1}$$

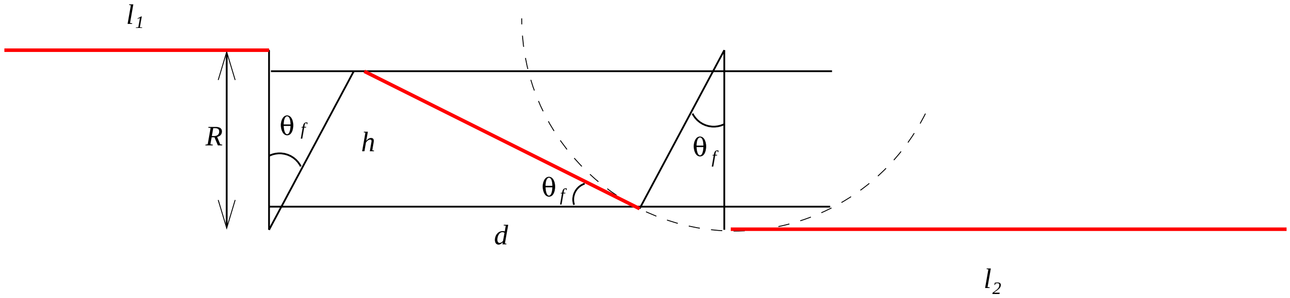

FIG. 2: Schematic of the first half of the low road.

where v_1 is the initial velocity of the ball. L is the total length of the track and can be written in terms of the parameters of the low road as

$$L = 2(l_1 + l_2 + d + 2R\sin\theta_f). \tag{2.2}$$

Calculating the time needed for the ball to traverse the low road is inherently more involved. As the track is symmetrical and the dissipative effects of rotational friction will be ignored in this theoretical treatment, we can calculate the time necessary to traverse half of the low road and multiply by a factor of two to obtain the total time. Furthermore, we divide the first half of the track into its five constituent parts, as can be seen in Fig. 2, and analyze each part in turn. The total time needed to traverse the low road, consequently, will be the sum of these individual times given by the relation

$$t_L = 2\sum_{i=1}^{5} t_i. \tag{2.3}$$

As was the case for the entire high road, the time necessary for the ball to traverse the upper horizontal section of the low road is the length of the segment divided by the velocity. This time is given by

$$t_1 = \frac{l_1}{v_1}. \tag{2.4}$$

We next analyze the lower horizontal section. In order to calculate the time needed for the ball to traverse the lower horizontal, we must first find the speed of the ball on this section. We find this speed by using the fact that the total mechanical energy is conserved, where again any energy loss due to rotational friction is neglected. Applying conservation of mechanical energy yields the expression

$$mgR + \frac{1}{2}mv_1^2 + \frac{1}{2}I\omega_1^2 = \frac{1}{2}mv_2^2 + \frac{1}{2}I\omega_2^2, \tag{2.5}$$

where the moment of inertia of a spherical ball is

$$I = \frac{2}{5}mR_b^2, \tag{2.6}$$

with R_b the proper radius of the ball. The angular velocity of the ball is proportional to the translational velocity and is

$$\omega = \frac{v}{R_e}, \tag{2.7}$$

where R_e is the effective radius of the ball, which differs from the proper radius. As the ball lies on two rails, the radius that connects the translational velocity to the angular velocity is the vertical distance between the point where the ball touches the rail and the center of the ball. See Appendix A for a more detailed discussion and derivation of this effective radius in terms of the proper radius.

Inserting Eqs. (2.6) and (2.7) into Eq. (2.5), we obtain the expression

$$mgR + \frac{1}{2}m\left[1 + \frac{2}{5}\left(\frac{R_b}{R_e}\right)^2\right]v_1^2 = \frac{1}{2}m\left[1 + \frac{2}{5}\left(\frac{R_b}{R_e}\right)^2\right]v_2^2. \tag{2.8}$$

Now, solving for the velocity of the ball on the lower horizontal in terms of the initial velocity is given by

$$v_2 = \sqrt{v_1^2 + 2g'R}\,, \tag{2.9}$$

where

$$g' \equiv g\left[1 + \frac{2}{5}\left(\frac{R_b}{R_e}\right)^2\right]^{-1} \tag{2.10}$$

plays the role of an effective gravitational acceleration and will be used throughout this paper. Having an expression for the velocity of the ball on the lower horizontal allows for a calculation of the time needed to traverse the lower horizontal in terms of the initial velociy. This time, in terms of the initial velocity of the ball and the parameters of the track, is given by the relation

$$t_2 = \frac{l_2}{\sqrt{v_1^2 + 2g'R}}. \tag{2.11}$$

B. The Upper Constant Radius of Curvature Section of the Low Road

The tangential acceleration of the center of mass of the ball on the upper constant radius of curvature section is dependent upon the angle subtended along the arc. Fig. 3 illustrates this scenario and includes a free-body diagram of the forces acting on the ball.

Newton's 2nd Law for translational motion in the centripetal and tangential directions are, respectively

$$mg\cos\theta - n = \frac{mv^2}{r} \tag{2.12}$$

$$mg\sin\theta - f_r = ma_{cm}, \tag{2.13}$$

where $r = R + R_e$ is the distance from the center of the radius of curvature of the track to the center of the spherical mass. f_r is the static friction acting by the rails on the ball that produces rotation, a_{cm} is the acceleration of the center of mass of the ball. Notice that our analysis assumes that the ball never leaves the track. This remains true so long as the speed of ball is less than $\sqrt{gr\cos\theta}$. For speeds larger than this, the normal force on the ball becomes zero and the ball leaves the track.

The acceleration of the center of mass is related to the angular quantities by

$$a_{cm} = r\ddot{\theta} = R_e\ddot{\phi}, \tag{2.14}$$

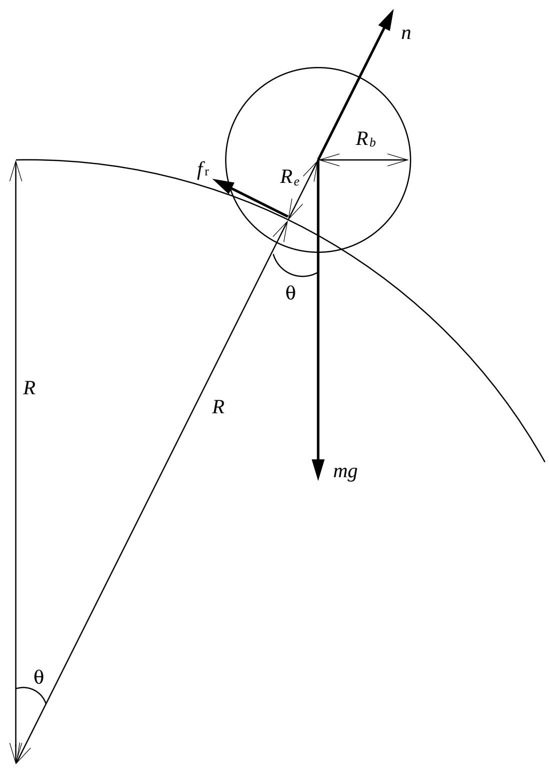

FIG. 3: Schematic of the ball traversing the upper constant radius of curvature section.

where $\ddot{\phi}$ is the angular acceleration of the ball about its center of mass. Newton's 2nd Law for rotational motion yields the expression

$$f_r R_e = \frac{2}{5} m R_b^2 \ddot{\phi}. \tag{2.15}$$

Substituting Eqs. (2.15) and (2.14) into Eq. (2.13) and rearranging, we arrive at the dynamical expression

$$\ddot{\theta} = \dot{\theta}\frac{d\dot{\theta}}{d\theta} = \frac{g'}{r}\sin\theta, \tag{2.16}$$

where we used the chain rule to connect the angular acceleration of the ball to its angular velocity. One can now separate variables and integrate to arrive at an expression for the angular velocity

$$\dot{\theta}^2(\theta) = \frac{v_1^2}{r^2} + \frac{2g'}{r}\left(1 - \cos\theta\right), \tag{2.17}$$

where we applied the boundary condition $\dot{\theta}(0) = v_1/r$ to obtain a value for the integration constant. The final velocity of the ball on the constant radius of curvature section takes the form

$$v(\theta_f) = \sqrt{v_1^2 + 2g'r(1 - \cos\theta_f)}, \tag{2.18}$$

which, consequently, will be the ball's initial speed on the diagonal section of the track.

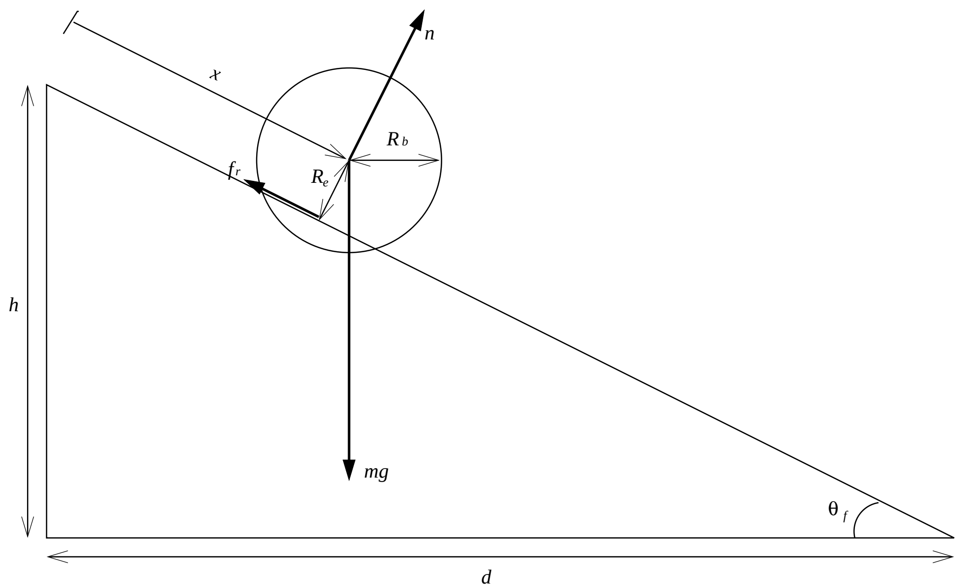

FIG. 4: Schematic of the ball traversing the diagonal section of the track.

Eq. (2.17) can be further separated and integrated. One arrives at an integral expression for the time spent on the constant radius of curvature section given by

$$t = \int_0^{\theta_f} \frac{r\,d\theta}{\sqrt{v_1^2 + 2g'r(1-\cos\theta)}}. \tag{2.19}$$

This integral is exactly solvable with the solution a hypergeometric function. To keep this analysis as simplistic as possible, we instead employ a small angle approximation and solve the integral perturbatively. To lowest order in θ

$$1 - \cos\theta \simeq \theta^2/2. \tag{2.20}$$

For our track, this approximation amounts to a maximum percent error, when $\theta = \theta_f$, of $\sim$1%. Using the above approximation, the integral yields an expression for the time spent on the constant radius of curvature section of the track

$$t_3 \simeq \sqrt{\frac{r}{g'}} \sinh^{-1} \quad \frac{\sqrt{g'r}}{v_1}\,\theta_f \quad . \tag{2.21}$$

C. The Diagonal Section of the Low Road

Applying Newton's 2nd Law to the ball on the diagonal in the parallel/perpendicular directions yields

$$n - mg\cos\theta_f \;=\; 0 \tag{2.22}$$
$$mg\sin\theta_f - f_r \;=\; m\ddot{x}, \tag{2.23}$$

where x is the location of the center of mass of the ball along the diagonal path, as shown in Fig. 4. Assuming that the ball does not slip on the diagonal, the coordinates x and ϕ are proportional and related by

$$\phi = \frac{x}{R_e}, \tag{2.24}$$

where ϕ is the angular displacement of the ball.

Newton's 2nd Law for rotational motion is again given by

$$f_r R_e = \frac{2}{5} m R_b^2 \ddot{\phi}. \tag{2.25}$$

Substituting Eqs. (2.24) and (2.25) into Eq. (2.23) and rearranging yields the dynamical expression

$$\ddot{x} = g' \sin\theta_f, \tag{2.26}$$

where the effective acceleration of gravity, g', was defined in Eq. (2.10).

As the acceleration of the ball on the incline is constant, we can invoke the kinematic equations

$$\begin{aligned} v &= v_i + a\Delta t \\ \Delta x &= v_i \Delta t + \frac{1}{2} a \Delta t^2 \\ v^2 &= v_i^2 + 2a\Delta x \end{aligned} \tag{2.27}$$

to find the time needed to traverse the diagonal section. The initial velocity, v_i, of the ball on the diagonal is known in terms of the initial velocity, v_1, of the ball at the beginning of the track. Eq. (2.18) yields

$$v_i = \sqrt{v_1^2 + 2g'r(1 - \cos\theta_f)}. \tag{2.28}$$

Plugging this into Eq. (2.27) gives the speed of the ball at the end of the diagonal section

$$v(x = \sqrt{h^2 + d^2}) = \sqrt{v_1^2 + 2g'[r(1 - \cos\theta_f) + h]}. \tag{2.29}$$

We now obtain an expression for the time to traverse the diagonal section of the track given by

$$t_4 = \frac{1}{g' \sin\theta_f} \left[\sqrt{v_1^2 + 2g'[r(1 - \cos\theta_f) + h]} - \sqrt{v_1^2 + 2g'r(1 - \cos\theta_f)} \right] \tag{2.30}$$

in terms of the parameters of the track and the initial velocity of the ball.

D. The Lower Constant Radius of Curvature Section of the Low Road

A derivation of the time for the ball to traverse the lower constant radius of curvature section closely parallels that of the upper constant radius of curvature section. Newton's 2nd law for translational motion in the centripetal and tangential direction are

$$n - mg\cos\theta = \frac{mv^2}{\tilde{r}} \tag{2.31}$$

$$mg\sin\theta - f_r = ma_{cm} \tag{2.32}$$

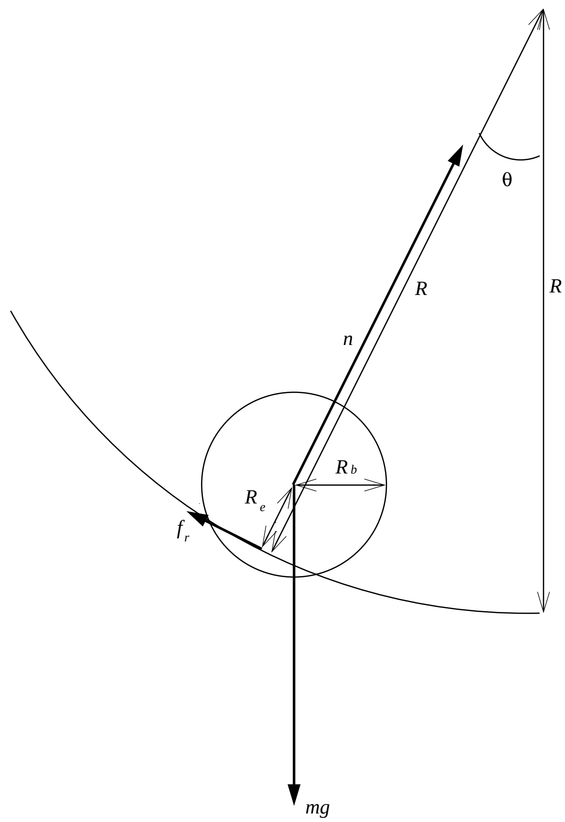

FIG. 5: Schematic of the ball traversing the lower constant radius of curvature section.

where $\tilde{r} = R - R_e$ is the distance from the center of the radius of curvature of the track to the center of the spherical mass. Notice that θ is decreasing as ϕ, the angular displacement of the ball about its center of mass, is increasing. Thus, our angular quantities are related by where

$$a_{cm} = R_e\ddot{\phi} = -\tilde{r}\ddot{\theta}. \tag{2.33}$$

Newton's 2nd law for rotational motion is of the form

$$f_r R_e = \frac{2}{5} m R_b^2 \ddot{\phi} \tag{2.34}$$

Inserting Eq. (2.34) for f_r into Eq. (2.32) and rearranging, one obtains an expression of the form

$$\ddot{\theta} = -\frac{g'}{\tilde{r}} \sin\theta \tag{2.35}$$

Now following a process similar to that of section IIB, we find the time needed to traverse the lower constant radius of curvature section is

$$t_5 \simeq \sqrt{\frac{\tilde{r}}{g'}}\ \sin^{-1}\left(\sqrt{\frac{g'\tilde{r}}{v_1^2 + 2g'R}}\ \theta_f\right), \tag{2.36}$$

where we set our limits of integration from $-\theta_f$ to 0, again used a small angle approximation, and applied the final boundary condition Eq. (2.9) for the velocity. The velocity expressed in Eq. (2.9) is that of the ball's as it traverses the lower horizontal section of the track.

E. Total Time Difference vs. Initial Speed

We can now calculate the total time difference between transits of the high and low road for a given initial velocity. The total time difference is defined as the time interval needed for the ball to traverse the high road minus the time interval needed for the ball to traverse the low road

$$\Delta t = t_H - t_L. \tag{2.37}$$

As the ball traversing the low road *always* wins the race, one expects Eq. (2.37) to be positive for all values of the initial velocity. In a previous subsection, it was found that the time for the ball to traverse the high road was given by

$$t_H = \frac{2(l_1 + l_2 + d + 2R\sin\theta_f)}{v_1}. \tag{2.38}$$

The time needed for the ball to traverse the low road is found by summing the time contributions from each of the five sections of the track and multiplying by a factor of two. The expression is of the form

$$\begin{aligned} t_L \;=\; 2\; & \frac{l_1}{v_1} + \frac{l_2}{\sqrt{v_1^2 + 2g'R}} + \sqrt{\frac{r}{g'}}\sinh^{-1}\frac{\sqrt{g'r}}{v_1}\theta_f + \sqrt{\frac{\tilde{r}}{g'}}\sin^{-1}\sqrt{\frac{g'\tilde{r}}{v_1^2 + 2g'R}}\,\theta_f \\ & + \frac{2}{g'\sin\theta_f}\; \sqrt{v_1^2 + 2g'[r(1-\cos\theta_f) + h]} - \sqrt{v_1^2 + 2g'r(1-\cos\theta_f)} \;. \end{aligned} \tag{2.39}$$

Although the above expressions are lengthy, it is important to notice that the time needed to traverse either track is simply a function of the initial velocity of the ball and the measurable parameters of the track.

III. EXPERIMENTAL SETUP

The high road/low road track was built as part of a senior research project in addition to the accompanying theoretical treatment presented in the aforementioned section and the data analysis presented in the next. A pair of rails were fashioned so that a steel ball nested on them as illustrated in Fig 7. The rails were constructed from quarter-inch steel rod. The shape of the lower track was engineered by bending the two rails to match a template laid out on a board. Trusses were built and welded vertically to the rails in order to support the tracks. Spacers were then welded at roughly 7 in. intervals in order to maintain an equal spacing between the rail pairs. A ramp was welded to one end of each track, which allows for a variety of speeds to easily be obtained. This is accomplished by simply varying the initial height of the ball on the ramp. The base of the track consists of a straight 2 in.×10 in.×10 ft. long piece of lumber. For support, holes were drilled into the wood to receive the ends of the trusses. An image of the track is displayed in Fig. 1. Table I shows the quantitative specifications of the track.

Measurements of the initial and final speed of the ball and the time interval needed for the ball to traverse a given track were made via Pasco Photogates and a Data Studio interface system. A photogate works by measuring the time interval that an infrared light in a given gate is blocked. From this measurement, Data Studio can then easily calculate the average speed of the object as it moves through a gate by comparing this time interval to

Track parameters	Symbol	Value
Total Length	L	3.00 m
Upper Horizontal	l_1	0.28 m
Lower Horizontal	l_2	0.91 m
Radius of Curvature	R	0.10 m
Rise of Diagonal	h	0.09 m
Run of Diagonal	d	0.24 m
Deviation Angle	θ_f	0.37 rad
Radius of Ball	R_b	0.0135 m
Effective Radius of Ball	R_e	0.0093 m
Radius of Rail	R_r	0.0033 m
Distance Between Rails	w	0.0179 m
Effective Gravitational Acceleration	g'	5.29 m/s^2

TABLE I: Parameters of Track

the flag length, which here equates to the diameter of the ball. Photogates can also measure the time interval between the activation of two successive gates. A set of photogates was placed on each track for a total of four gates. One photogate was positioned at the start of each track and was used to obtain the initial speed of the ball. A second photogate was positioned at the end of each track. This second photogate yielded the final speed of the ball and allowed for a measurement of the total time interval needed for the ball to traverse a given track. The gates were carefully positioned to ensure that both sets spanned equal horizontal distances. It was also important that the gates were placed at the same height above the track so to be tripped by the same relative location of the ball. For an accurate flag length, the infrared light in the photogate obviously needed to align with the ball at its widest point.

The total drag due to the rotational friction of the ball-track interaction and the velocity-dependent air resistance results in a slowing of the ball as it traverses the track. Hence, the speed of the ball as measured by the second photogate can be significantly smaller than that measured by the first. For smaller speeds, this slowing plays a more significant fractional role. As our theoretical determination of the time difference, as summarized in Eqs. (2.37)-(2.39), neglected the dissipative effects of friction, our data is expected to deviate from the theoretical curve. To account for the dissipative effects, we use the average speed, as obtained from the measured initial and final speeds, as numerical input for v_1 in the theoretical curve displayed in Fig. 6. It should be noted that in the limit of vanishing frictional effects, the average speed and v_1 become equivalent.

IV. EXPERIMENTAL PROCEDURE & DISCUSSION

Data from the experiment was first collected on the low road. The desired range of speeds was probed by releasing the ball from rest at a chosen initial ramp height. The ball was then allowed to traverse the track. Data Studio displayed values for the initial and final speeds and the time interval necessary for the ball to traverse the track. From this data, the average speed could be calculated for a given time interval. We repeated several low road track runs with the intent of covering the widest range of speeds possible, while obtaining proper representation of the intermediate speeds. The minimum speed recorded corresponded to the ball that just finished the track with vanishing speed; the maximum corresponded to that where the ball just remained in contact with the track. For speeds above this maximum, the 'hum' of the ball on the track would lessen as the ball traversed the upper constant radius of curvature section thus coming unattached. These data points were discarded as our theoretical analysis demands that the ball stay in contact with the track.

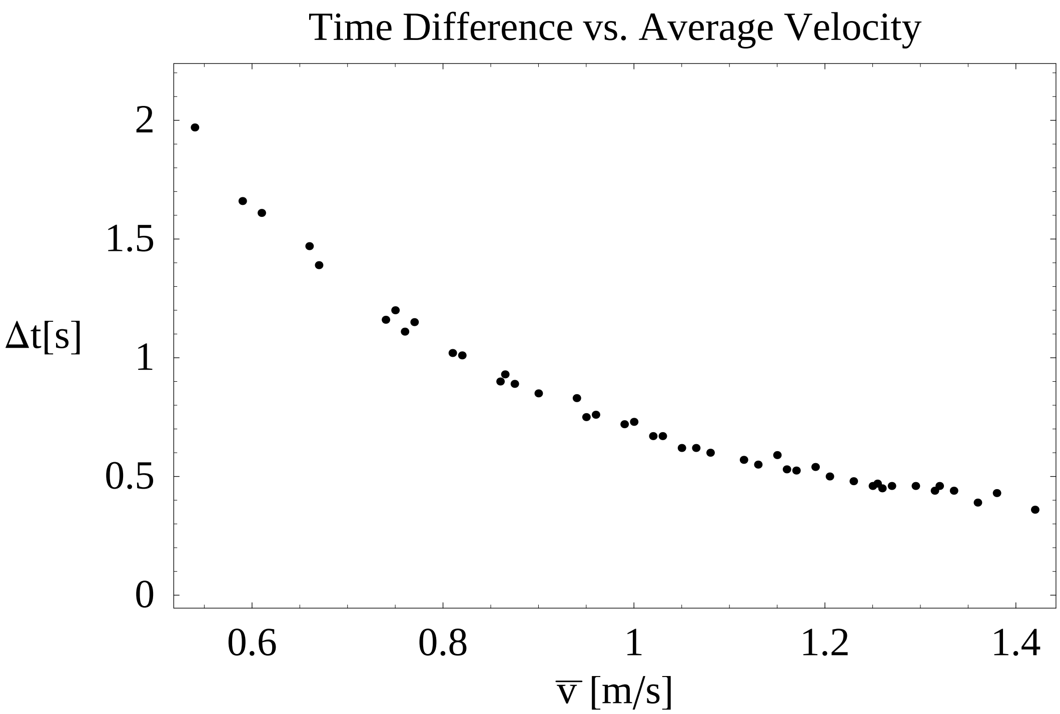

FIG. 6: Plot of the time difference, $\Delta t = t_H - t_L$, versus the average velocity of the ball, $\bar{v}$.

We then repeated the above process for the high road. The average speed for a given run was again calculated and compared to those of the low road. If this average speed matched any of those for runs on the low road (to within 0.005 m/s), then the time difference was calculated and the data point was recorded. Fig. 6 displays the data and the theoretical prediction, as calculated in section II, with the average speed used as numerical input for v_1.

In general, the high road/ low road demonstration serves as an excellent teaching tool of classical dynamics. The results of the demonstration are counterintuitive and can be used to engage the students who are often surprised of the outcome. A detailed theoretical analysis can proceed and be used to cover much of Newtonian mechanics. Finally, measurements of the time difference as a function of the average speed for a high road/ low road demonstration can easily be made in the laboratory and be used to help reinforce the concepts and theory previously encountered.

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APPENDIX A: EFFECTIVE RADIUS

To find the effective radius, R_e, we need to first calculate the angle of contact, α, between the ball and the rails. For an illustration of these quantities and the cross section of the ball and rails, see Fig. 7. As can be witnessed from the figure

$$2R_b \cos\alpha = w + 2R_r(1-\cos\alpha) \tag{A1}$$

where w is the distance between the rails and R_b and R_r are the radii of the ball and rail, respectively. Solving for the angle and using basic trigonometry, one finds

$$\alpha = \cos^{-1}\left(\frac{w+2R_r}{2(R_b+R_r)}\right) = \sin^{-1}\left(\sqrt{1-\frac{1}{4}\left(\frac{w+2R_r}{R_b+R_r}\right)^2}\right) \tag{A2}$$

In this study it is advantageous to find the ratio of the radius of the ball to the effective

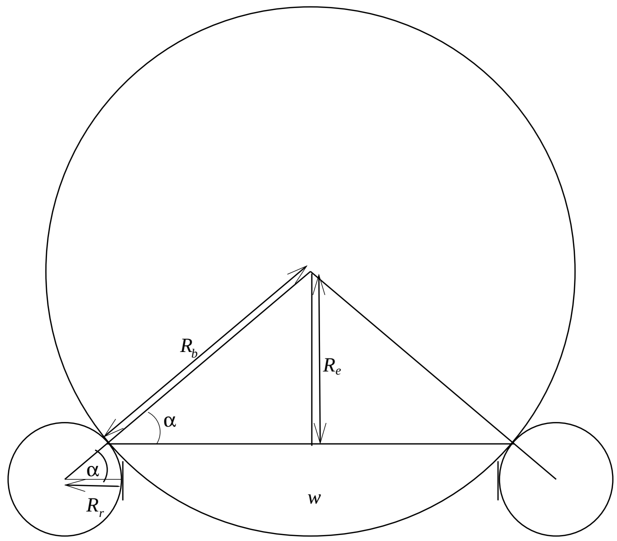

FIG. 7: Schematic of the cross section of the ball and the rails of the track.

radius, thus determining a unit-less scaling factor. Again from basic trigonometry, we find

$$\frac{R_b}{R_e} = \frac{1}{\sin\alpha} = \left(1 - \frac{1}{4}\left(\frac{w + 2R_r}{R_b + R_r}\right)^2\right)^{-1/2} = 1.46 \tag{A3}$$

to three significant figures where we used the data from Table I.