THE IDEA OF MODELING

One of the most powerful and elegant ways to represent simple relationships between two or more quantities in a physical system is by using mathematical equations. Sometimes, when examining a graph of data representing how one quantity changes with another, you'll find that the data points tend to lie along a straight line or a fairly smooth curve.

Fitting a Straight Line

The horizontal distance moved by a baseball after it leaves the pitcher's hand might appear to increase more or less linearly as shown in Figure E.1.

<table>
<thead>
<tr>
<th>Time(s)</th>
<th>Distance(m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.0</td>
</tr>
<tr>
<td>0.47</td>
<td>10.0</td>
</tr>
<tr>
<td>0.64</td>
<td>15.0</td>
</tr>
<tr>
<td>0.98</td>
<td>20.0</td>
</tr>
<tr>
<td>1.41</td>
<td>25.0</td>
</tr>
</tbody>
</table>

In general, the equation for a straight line is represented by the equation

\[ y = mx + b \]

where \( x \) represents an independent variable plotted on the horizontal axis and \( y \) represents a dependent variable plotted on the vertical axis. The \( m \) represents the slope of the graph and the \( b \) represents the \( y \)-intercept of the graph (i.e., the value of \( y \) when \( x \) is zero). Thus, we would expect that we could model the horizontal motion of the pitched baseball being described in the graph shown in Fig. E.1 by using the equation \( \text{Distance} = m \times \text{time} + b \). In this case the \( y \)-intercept, \( b \), is obviously zero. The data points don't seem to lie exactly along a line, just approximately. What is the best value for the slope, \( m \)? One way to estimate a reasonable value is to plot Distance vs. Time for a number of different values of \( m \) and see which line seems to pass closest to most of the data points. Various possible slopes are shown in the graph below.

The equation for the line on the graph that looks pretty good turns out to be

\[ \text{Distance (m)} = (19.5 \text{ m/s}) \times \text{Time (s)} \]

Thus, the slope of a pretty good line is 19.5 meters per second and its \( y \)-intercept is 0.0 m. This process of looking at the shape of a graph and finding an idealized equation that more or less fits the data is called mathematical modeling. In our simple linear, or straight line, example, the equation that results is a very elegant shorthand for describing the horizontal progress of a particular pitched baseball as it travels. The purpose of mathematical modeling is to find equations and appropriate constants that seem to best describe a given physical system.
Modeling a “Curve”

Not all data points lie along a straight line. For example, let’s consider some idealized data on the height of a miniature rocket as a function of time. These data are shown in Table E.1.

<table>
<thead>
<tr>
<th>Altitude (ft)</th>
<th>Time (s)</th>
<th>Time² (s²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>32.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>128.0</td>
<td>2.0</td>
<td>4.0</td>
</tr>
<tr>
<td>288.0</td>
<td>3.0</td>
<td>9.0</td>
</tr>
</tbody>
</table>

A plot of altitude vs. time would yield a curve as shown in Figure E.3.

This graph looks a bit like one of the parabolas we all learned about in algebra classes. The general equation for a parabola is given by the expression

\[ y = ax^2 + bx + c \]

where \( a \) and \( b \) are constants. If we were to attempt to find an equation that seems to describe the altitude of our rocket as a function of time, we might guess that the expression would be something like

\[ \text{Altitude} = a_2 \times \text{Time}^2 + a_1 \times \text{Time} + a_0 \]

since time can be assigned as the independent variable plotted along the x-axis and the altitude can be assigned the role of the dependent variable plotted along the y-axis. To do mathematical modeling you would need to plot the theoretical data for different values of \( a_1 \), \( a_2 \), and \( a_0 \) until a theoretical curve and the experimental curve match pretty well. The process can be simplified a bit because someone with experience would recognize immediately that a parabola of the type that fits the rocket data would have \( a_1 = 0 \) and \( a_0 = 0 \). Thus, only a “best” value for \( a_2 \) is needed.

Suppose you graphed some other data that doesn’t look like either a parabola or a straight line. and you don’t know the exact mathematical relationship that can be used to fit your data. For example, a graph of the temperature of a cup of coffee that is cooling over time does not look like either a parabola or a straight line. In that case you need to guess the functional form of your curve using a knowledge of physics theory or a familiarity with the shapes of common functions. You might try powers of \( x \); or \( e^x \), or perhaps \( \ln(x) \) in some cases.

MATHEMATICAL MODELING WITH AN EXCEL SPREADSHEET

Unfortunately, it is very tedious to guess an equation because after guessing the form of the equation, you also need to find good values for the constants in the equation and then plot graphs for each guess. Fortunately, an Excel worksheet with its dynamic graphing capability can be used to make the modeling process much easier.

An Excel Modeling Tutorial (MODTUT.XLS) has been developed to help you learn about the process of spreadsheet modeling. Each time you perform simple one-dimensional mathematical modeling, you can open a Worksheet file and set it up so that one column of data represents the experimental data points and another column of data represents the proposed theoretical data. These columns can be linked to a graph. The first time you attempt to do mathematical modeling you should open (MODTUT.XLS) the Modeling Tutorial. In this tutorial various notes and messages have been included to help you get started with the modeling process.

A portion of the Modeling Tutorial worksheet is shown in Fig. E.4. Open up the Modeling Tutorial spreadsheet and enter a mathematical relationship in the first cell in the y-theory column that fits the sample experimental data given.
After modeling the data in the tutorial spreadsheet, you can open a new worksheet and try to model the data shown in Figure E.1 describing the horizontal distance a pitched baseball moves over time. Do you agree that the mathematical equation is a linear one? Is 19.5 m/s a good estimate for the slope?

BECOMING FAMILIAR WITH CURVE SHAPES FOR DIFFERENT FUNCTIONS

If data seems to lie along a certain type of curve or if a theory suggests that data ought to lie along a certain curve, then you can make an intelligent guess about the equation needed for a mathematical model. To make such a guess, it is helpful to know what the shapes of curves associated with various mathematical expressions are like. For example, what does \( y = A \sin (Bx) \) look like for various values of \( A \) and \( B \)? Creating graphs of various types of functional relationships is a good way to begin to recognize the general shapes of curves. A special Excel toolset called the WPtools for Excel has been created to allow you to produce graphs of various functions easily.

Using the Graphing Tool to Display Common Functions Graphically:

You can use the Excel graphing tool to generate a graphical library of mathematical relations that frequently occur in physics. Some functions that can be plotted include:

1. \( y = mx + b \)
2. \( y = x^2 \)
3. \( y = x^P \) where \( P = 1, 2, 3 \ldots 98 \)
4. \( y = e^x \)
5. \( y = 1/x^P \) where \( P = 1, 2, 3 \ldots 98 \)
6. \( y = 1/e^x \)
7. \( y = x \)
8. \( y = \sqrt{x} \)
9. \( y = \ln(x) \)
10. \( y = A \sin (B \times x) \) for \( A = 1, 2, 3 \) and \( B = 1, 2, 3 \), etc.

To use the graphing tool, open an Excel Worksheet and enter values in an \( x \) column and values in a \( y \) column generated by the mathematical equation of interest. (See Appendix A for details on using the WPtools for Excel for Graphing.)

Let us carry out a systematic investigation of several sequences of functions.

a. Turn on your computer and open an Excel Worksheet file.

b. Title the first column \( x \), and enter about 20 equally spaced values of \( x \) of interest to you into the \( x \) raw column. For example, if you are interested in functions of \( x \) where \( x \) varies between 0 and 10 you could make a column of numbers such as 0.0, 0.5, 1.0, 1.5 \ldots 10.0.

c. Title the second column \( y \). Select the first cell in the \( y \) column. Then choose the formula you wish to use to transform your data. You may choose to take the natural log of your \( x \)-data, or see what happens when you raise it to a power. Once you've entered the formula or function of your choice, you can Fill Down into the rest of the \( y \)-raw column.

d. Finally, plot your raw data by selecting your \( x \) and \( y \) data columns and clicking on the graphing tool that is on the left-hand side of the custom toolbar at the bottom of the screen.

You should carry out a systematic investigation of several sequences of functions using the graphing tool. To record results, you should sketch the shape of each graph in your Lab Notebook or Activity Guide. Please label the axes for each graph carefully and note how the curves change as you proceed through a sequence of functions.
1. *Rapidly growing curves:* Try the functions numbered 2 to 4, and note how they become more “abrupt” in their departure from the $x$-axis.

2. *Rapidly dying curves:* Try the functions numbered 5 and 6.

3. *Slowly growing curves:* Finally, try functions numbered from 7 to 9. Note that the logarithm approaches negative infinity near $x = 0$.

All these relationships occur in some branch of physics. We hope you will find this exercise useful in later mathematical modeling activities. Indeed, now that you know how to use the WPtools for Graphing, you can call up your graphical library of functions at will. If you can guess the functional form of a physical relationship by viewing its graph, you can often develop a mathematical model to describe it.