APPENDIX F
UNCERTAINTY PROPAGATION—UNCERTAINTIES AFTER CALCULATIONS

Maids to bed and cover coal;
Let the mouse out of her hole;
Crickets in the chimney sing;
Whilst the little bell doth ring:
If fast asleep, who can tell
When the clapper hits the bell?

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UNCERTAINTIES COMPOUND

Imagine what would happen if you were trying to determine the area of a rug using an old, warped stick marked in centimeters from 0 to 300. You find the length of one side of the rug to be 200 cm, but since the measuring stick could have shrunk or expanded, you attach a 5-mm or 0.5-cm uncertainty to your measurement and write down 200 ± 0.5 cm for the length of one side. You make a mental note that this is a (0.5/200) × 100% = 0.25% percentage uncertainty. Now you go to measure the other side, and, lo and behold, you find that the rug appears to be square—you get 200 cm again, but you attach the same uncertainty to it: 200 ± 0.5 cm.

Let’s say you want to sell this rug to a wise old Armenian rug merchant, who asks you exactly how certain you are of its area. We will draw a diagram of the situation below.

The picture shows the rug with the best estimate of its dimensions, 200 × 200 cm. It also shows as white strips (greatly exaggerated in size!) the “extra area” that would result if each side was increased in length by the full amount of the uncertainty in the measurement of the length of a side. These two white strips contribute an extra area of 200 cm². So the rug’s area is uncertain by that amount. The area should be quoted as 40,000 ± 200 cm². We are ignoring the small black overlap region of area (0.25 cm²) since it is the square of an already small uncertainty in length.

Now comes the important point: the fractional uncertainty of the area of the rug is (200/40,000) × 100% = 0.5%. We have gone from a 0.25% uncertainty in the length of a side of the rug to a 0.5% uncertainty in the rug’s area! Although the wise old Armenian rug merchant might not complain, things are worse by a factor of two!

The point of this parable is this: when numbers that are themselves uncertain are used in a calculation to obtain a final result (squaring the length of one side of a square to get an area, for example) the final result is very often more uncertain, in percentage terms, than the original numbers were!

More specifically, if

\[ y = x^2 \]

then it can be shown that
{percentage uncertainty in $y$} = 
2 \times \{percentage uncertainty in $x$\}

We will outline the general theory behind all this, but you should keep in mind that percentage uncertainties tend to increase as new quantities are calculated. Frequently we start with measurements that we think are fair to good, but we can be unpleasantly surprised by the uncertainty in the final calculated result!

**THE PROPAGATION OF UNCERTAINTIES**

More often than not, the final result of an experiment involves the measurement of several numbers that are then combined in some formula. How do the uncertainties in each of these numbers combine to produce the uncertainty in the final result? The answer for the general case is a bit complex, but if we can assume that the uncertainties in each of the “input” numbers in the formula are unrelated (as is often true), we can use the simple rules that will be discussed below. *Unrelated uncertainties* means that the size of the uncertainty in one quantity does not directly affect the size of the uncertainty in any of the other input quantities.

Let us assume that our final result is $R = x + y + z$, and that we can estimate the uncertainties in $x$, $y$, and $z$. Then we have the following relationship between the uncertainty in the calculated result and those in $x$, $y$, and $z$:

- $\sigma_X = \text{Uncertainty in } x$
- $\sigma_Y = \text{Uncertainty in } y$
- $\sigma_Z = \text{Uncertainty in } z$
- $\sigma_R = \text{Uncertainty in } R$

$$\sigma_R = \sqrt{\sigma_X^2 + \sigma_Y^2 + \sigma_Z^2}$$

If, instead, $R = x + y - z$, the above rule still holds. In general, we say that for results involving sums and differences, uncertainties add “in quadrature,” meaning as a sum of squares.

If our calculations involve products and quotients, the method for determining the resulting uncertainty is more complex. In such a case it turns out that the fractional uncertainties, defined as the uncertainty divided by the best estimate of the value, add in quadrature. For example if the result is the product of $x$ and $y$ divided by $z$, so that $R = x y/z$ then:

$$\sigma_R = \frac{1}{R} \sqrt{\left(\frac{\sigma_x}{x}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2 + \left(\frac{\sigma_z}{z}\right)^2}$$

Another common situation is when $R$ is a function of some one variable, $R = f(x)$. Then:

$$\sigma_R = \left| \frac{df}{dx} \right| \sigma_X$$

Here the vertical bars indicate “absolute value”—so we are ignoring the sign of the derivative. The derivative is evaluated at the best estimate of $x$. A simple example of this is when the result is the product of a constant $A$ of $x$ so $R = Ax$. Then the above formula tells us that the uncertainty in $R$ is $A$ times the uncertainty in $x$:

$$\sigma_R = |A| \sigma_X$$

A more complex example would be the case where $R = \ln(x)$, where $\ln$ indicates the natural logarithm. Since the derivative of the natural logarithm is $1/x$, our rule then becomes

$$\sigma_R = \left| \frac{1}{x} \right| \sigma_X$$

We have presented these rules without derivation, but we hope that you find them at least plausible. An excellent discussion of these and many other related topics that we cannot hope to cover in this Appendix is given in the book *An Introduction to Error Analysis—The Study of Uncertainties in Physical Measurements* by John R. Taylor (University Science Books, Mill Valley, CA, 1982). The interested student is strongly encouraged to refer to this little book.