UNIT 13: ROTATIONAL MOMENTUM AND TORQUE AS VECTORS

In this stop-action photo a racing cyclist is moving at a high speed. It is very easy for this cyclist’s center of mass to be to one side or the other of his balance points where the wheels touch the ground. If the bicycle wheels are not spinning, he will topple over more easily than if they are spinning. Why? What is it about the action of the wheels that makes him appear to defy the laws of gravity? In this unit you will learn about how the vector relationship between torque and rotational momentum change can help stabilize the cyclist.
UNIT 13: ROTATIONAL MOMENTUM AND TORQUE AS VECTORS

Experiment is the ultimate touchstone throughout good science, whether it comes at the beginning as a gathering of empirical facts or at the end in the final tests of a grand conceptual scheme.

Eric M. Rogers
Physics for the Inquiring Mind

OBJECTIVES

1. To understand the definitions of torque and rotational momentum as vector quantities.

2. To understand the mathematical properties and some applications of the vector cross product.

3. To understand the relationship between torque and rotational momentum.

4. To understand the Law of Conservation of Rotational Momentum.
13.1. OVERVIEW

This unit presents a consolidation and extension of the concepts in rotational motion that you have already studied. In the last unit you studied the relationships between rotational and linear quantities such as position and angle, linear velocity and rotational velocity, linear acceleration and rotational acceleration, and force and torque. You did this without taking into account the fact that these quantities have directions associated with them that can be described by vectors. We will discuss the vector nature of rotational quantities and, in addition, define a new vector quantity called rotational momentum, which is the rotational analog of linear momentum.

Rotational momentum and torque are special vectors because they are the product of two other vectors—a position vector and a force or linear momentum vector. In order to describe them we need to introduce a new type of vector product known as the vector cross product. We will explore the definition and unique nature of the vector cross product used to define torque and rotational momentum.

We will also study the relationship between torque and rotational momentum as well as the theoretical basis of the Law of Conservation of Rotational Momentum. At the end of this unit you will experience the effects of rotational momentum conservation by holding masses in your hands and pulling in your arms while rotating on a platform. You will be asked to calculate your rotational inertia with your arms in and with your arms out by making some simplifying assumptions about the shape of your body.
TORQUE, VECTORS, AND ROTATIONAL MOMENTUM

13.2. OBSERVATION OF TORQUE WHEN F AND r ARE NOT ⊥

In the last unit, you “discovered” that if we define torque as the product of a lever arm and perpendicular force, an object does not rotate when the sum of the torques acting on it adds up to zero. However, we didn’t consider cases where \( \vec{F} \) and \( \vec{r} \) are not perpendicular, and we didn’t figure out a way to tell the direction of the acceleration resulting from a torque. Let’s consider these complications by generating torques with spring balances and a lever arm once more. For this activity you’ll need:

- 1 horizontal pivot
- 1 clamp stand (to hold the pivot)
- 2 spring scales
- 1 ruler
- 1 protractor

Recommended Group Size: 2
Interactive Demo OK?: N

13.2.1. Activity: Torque as a Function of Angle

a. Suppose you were to hold one of the scales at an angle of 90° with respect to the lever arm, \( \vec{r}_h \), and pull on it with a steady force. Meanwhile you can pull on the other scale at several angles other than 90° from its lever arm, \( \vec{r}_{app} \), as shown below. Would the magnitude of the balancing force be less than, greater than, or equal to the force needed at 90°? What do you predict? Explain.

Fig. 13.2.
UNIT 13: ROTATIONAL MOMENTUM AND TORQUE AS VECTORS

b. You should determine exactly how the non-perpendicular forces compare to that needed at a 90° angle. Determine forces for at least four different angles and figure out a mathematical relationship between $F$, $r$, and $\theta$. Set up a spreadsheet to do the calculations shown in the table below. **Hint:** Should you multiply the product of the measured values of $r$ and $F$ by $\sin \theta$ or by $\cos \theta$ to get a torque that is equal in magnitude to the holding torque?

<table>
<thead>
<tr>
<th>Holding Torque</th>
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</thead>
<tbody>
<tr>
<td>$r_i$ (m)</td>
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<table>
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<tr>
<th>Applied Torque</th>
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<tbody>
<tr>
<td>$r_{app}$ (m)</td>
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The activity you just completed should give you a sense of what happens to the magnitude of the torque when the pulling force, $\vec{F}$, is not perpendicular to the vector, $\vec{r}$, from the axis of rotation. But torque is a vector and has both a magnitude and direction associated with it. How do we define the direction of the torque vector? Let’s consider the directions we might associate with rotational velocity and torque in this situation.
13.2.2. Activity: Rotational Rotation, Torque, and Direction

a. Suppose a particle is moving around in a circle with a rotational velocity that has a magnitude \( \omega \) associated with it. Suppose the plane of motion is perpendicular to the line of sight of each of two observers as shown in Figure 13.3. According to observer #1, does the particle appear to be moving clockwise or counterclockwise? How about the direction of the particle’s motion according to observer #2?

![Figure 13.3](image)

b. Is the clockwise vs. counterclockwise designation a good way to determine the direction associated with \( \omega \) in an unambiguous way? Why or why not?

c. Can you devise a better way to assign a direction to rotational velocity?

d. Similar consideration needs to be given to torque as a vector. Can you devise a rule to assign a direction to a torque? Describe the rule.

13.3. DISCUSSION OF THE VECTOR CROSS PRODUCT

An alternative to describing rotations as clockwise or counterclockwise is to associate a positive or negative vector with the axis of rotation using an arbitrary but well-accepted rule called the right-hand rule. By using vectors we can describe separate rotations of many body systems all rotating in different planes about different axes.

By using this vector assignment for direction, rotational velocity and torque can be described mathematically as “vector cross products.” The vector cross product is a strange type of vector multiplication worked out years ago by mathematicians who had never even heard of rotational velocity or torque.
The peculiar properties of the vector cross product and its relationship to rotational velocity and torque are explained in most introductory physics textbooks. The key properties of the vector that is the cross product of two vectors \( \vec{r} \) and \( \vec{F} \) are:

1. The magnitude of the cross product is given by \( rF \sin \theta \) where \( \theta \) is the smallest angle between the two vectors. Note that the term \( F \sin \theta \) represents the component of \( \vec{F} \) along a line perpendicular to the vector \( \vec{r} \).

2. The cross product of two vectors \( \vec{r} \) and \( \vec{F} \) is a vector that lies in a direction \( \perp \) to both \( \vec{r} \) and \( \vec{F} \) and whose direction is given by the right hand rule. Extend the fingers of your right hand in the direction of the first vector \( \vec{r} \) and then rotate your fingers towards the second vector \( \vec{F} \) and your thumb will then point in the direction of the resultant cross product \( \vec{\tau} \).

These properties of the cross product are pictured below.

Fig. 13.4. Diagram of the vector cross product.

The spatial relationships between \( \vec{r} \), \( \vec{F} \), and \( \vec{\tau} \) are very difficult to visualize. In the next activity you can connect some thin rods of various sizes to each other at angles of your own choosing and make some “vector cross products.” For this activity you will need the following items:

- 3 connectors
- 3 wooden skewers
- Styrofoam balls, 1-inch diameter, or toothpicks and dried peas soaked in water
- 1 protractor

13.3.1. Activity: Making Models of Vector Cross Products

a. Pick out rods of two different lengths and connect them at some angle you choose. Consider one of the rods to be the \( \vec{r} \) vector and the other to be the \( \vec{F} \) vector. Measure the angle \( \theta \) and the lengths of \( \vec{r} \) and \( \vec{F} \) in meters. Then compute the magnitude of the cross product as \( rF \sin \theta \) in newton - meters (N⋅m). Show your units. Note: You should assume that the magnitude of the force in newtons is represented by the length of the rod in meters so that
\[ |\vec{\tau}| = |\vec{r} \times \vec{F}| = rF \sin \theta \] where \( r \) and \( F \) are the magnitudes of the vectors \( \vec{r} \) and \( \vec{F} \) respectively.

b. Attach a “cross product” rod perpendicular to the plane determined by \( \vec{r} \) and \( \vec{F} \) with a length of \( rF \sin \theta \). Sketch the location of \( \vec{F} \) relative to \( \vec{r} \) in the space below. Show the direction and magnitude of the resultant torque \( \vec{\tau} \). Finally, show your Styrofoam and skewer cross product model to an instructor, teaching assistant, or fellow student for confirmation of its validity.

![Diagram of cross product](image)

Fig. 13.5.

c. In the diagrams below the vectors \( \vec{r} \) and \( \vec{F} \) lie in the plane of the paper. Calculate the torques for the following two sets of \( \vec{r} \) and \( \vec{F} \) vectors. In each case measure the length of the \( \vec{r} \) vector in meters and assume that the length of the \( \vec{F} \) vector in cm represents the force in newtons. Use a protractor to measure the angle, \( \theta \), between the extension of the \( \vec{r} \) vector and the \( \vec{F} \) vector. Calculate the magnitude of the torques. Place the appropriate symbol to indicate the direction of the torque in the circle as follows:

\[ r = \underline{\phantom{000}} \text{m} \quad r = \underline{\phantom{000}} \text{m} \]

\[ F = \underline{\phantom{000}} \text{N} \quad F = \underline{\phantom{000}} \text{N} \]

\[ \theta = \underline{\phantom{000}} \text{rad} \quad \theta = \underline{\phantom{000}} \text{rad} \]

\[ \tau = \underline{\phantom{000}} \text{N}\cdot\text{m} \quad \tau = \underline{\phantom{000}} \text{N}\cdot\text{m} \]
13.4. MOMENTUM AND ITS ROTATIONAL ANALOG

Once we have defined the properties of the vector cross product, another important rotational vector is easily obtained, that of rotational momentum relative to an axis of rotation.

13.4.1. Activity: Rotational and Linear Momentum

a. Write the rotational analogs of the linear entities shown. Note: Include the formal definition (which is different from the analog) in spaces marked with an asterisk (*). For example, the rotational analog for velocity is rotational velocity \( \vec{\omega} \) and the definition of its magnitude is \( \omega \equiv |d\theta / dt| \) rather than \( v/r \).

<table>
<thead>
<tr>
<th>Linear entity</th>
<th>Rotational</th>
<th>Analog definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X ) (position)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \vec{v} ) (velocity)</td>
<td>*</td>
<td></td>
</tr>
<tr>
<td>( \vec{a} ) (acceleration)</td>
<td>*</td>
<td></td>
</tr>
<tr>
<td>( \vec{F} ) (Force)</td>
<td>*</td>
<td></td>
</tr>
<tr>
<td>( m ) (mass)</td>
<td>*</td>
<td></td>
</tr>
<tr>
<td>( \vec{F} = m\vec{a} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b. What do you think will be the rotational definition of rotational momentum in terms of the vectors \( \vec{r} \) and \( \vec{p} \)? Hint: This is similar mathematically to the definition of torque and also involves a vector cross product. Note that torque is to rotational momentum as force is to momentum.

c. What is the rotational analog in terms of the quantities \( I \) and \( \vec{\omega} \)? Do you expect the rotational momentum to be a vector? Explain.

d. Summarize your guesses in the following table.

<table>
<thead>
<tr>
<th>Linear equation</th>
<th>Rotational equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \vec{p} \equiv m\vec{v} ) [definition in terms of ( \vec{r} &amp; \vec{p} )]</td>
<td>( \vec{L} \equiv )</td>
</tr>
<tr>
<td>( \vec{p} = m\vec{\omega} ) [analog using ( I, \vec{\omega} )]</td>
<td>( \vec{L} = )</td>
</tr>
</tbody>
</table>
13.5. OBSERVING A SPINNING BICYCLE WHEEL

If a bicycle wheel is spinning fairly rapidly, can it be turned easily so that its axis of rotation points in a different direction? If its axis is perfectly vertical while it is spinning, will the wheel fall over? Alternatively, does it fall over when the wheel is not spinning? To make these observations you will need:

- 1 bicycle or Soap Box Derby wheel* (mounted on an axle)
- 1 string (to wrap around the rim of the wheel)

| Recommended Group Size: | 4 | Interactive Demo OK?: | Y |

13.5.1. Activity: Is Spinning More Stable?

a. Do you expect it to take more torque to change the axis of rotation of a wheel that is spinning rapidly or one that is spinning slowly? Or do you expect the amount of torque to be the same in both cases? Explain.

b. Hold the wheel axis along a vertical line while the wheel is not spinning and change the axis from a vertical to a horizontal direction. Describe the “torque” it takes qualitatively.

c. Have someone help you get the wheel spinning rapidly while you hold the axle vertical. While the wheel is spinning, change the axis to the horizontal direction. Describe the “torque” it takes qualitatively. How does the torque compare to that needed to change the direction of the axis of rotation of the wheel when it is not spinning? Did you observe what you expected to observe?

* Available from International Soap Box Derby, Inc., P.O. Box 7233, Akron, OH 44306.
d. Does the magnitude of the rotational velocity vector change as you change the axis of rotation of the wheel? Does its direction change? Does the rotational velocity vector change or remain the same? Explain.

e. Does the rotational momentum vector change as you change the axis of rotation of the spinning wheel? Why or why not?

f. If possible, use your answer to part e. to “explain” what you observed in part c.

13.6. TORQUE AND CHANGE OF ROTATIONAL MOMENTUM

Earlier in this course you applied a very brief force along a line through the center of mass of a rolling cart. Do you remember how it moved? What happened when you applied a gentle but steady force along a line through the center of mass of the cart? Let’s do analogous things to a disk that is free to rotate on a relatively frictionless bearing, with the idea of formulating laws for rotational motion that are analogous to Newton’s laws for linear motion. For this observation, you will need:

- 1 rotational apparatus*
- 1 clamp stand (to mount the system on)
- 1 string
- 1 mass set, 20 g and 50 g

Recommended Group Size: 4  Interactive Demo OK?: Y

*This should include a rotating disk with a spool or spindle attached to an axle that can rotate freely, such as the PASCO Rotational System or the PASCO Rotary Motion Sensor with the Chaos Accessory attached to it.
Figure out how to use a system like that shown in Figure 13.7 to observe the motion of the disk under the influence of a brief torque and a steady torque. In describing the Laws of Rotational Motion, be sure to consider vector properties and take both the magnitudes and directions of the relevant quantities into account in your wordings.

13.6.1. Activity: Applied Torques and Resultant Motion

a. What happens to the rotational velocity and hence the rotational momentum of the disk before, during, and after the application of a brief torque? State a First Law of Rotational Motion (named after yourself, of course) in terms of torques and rotational momenta. **Hint:** Newton’s first law states that the center of mass of a system of particles or a rigid object that experiences no net external force will continue to move at constant velocity.

The Rotational First Law in words:

The Rotational First Law as a mathematical expression:

b. What happens to the magnitude and direction of the rotational velocity (and hence the rotational momentum) of the disk during the application of a steady torque? How do they change relative to the magnitude and direction of the torque? If possible, give a precise statement of a Second Law of Rotational Motion relating the net torque on an object to its change in rotational momentum. **Note:** Take both magnitudes and directions of the relevant vectors into account in your statement.
**Hint:** Newton’s second law of motion states that the center of mass of a system of particles or rigid object that experiences a net external force will undergo an acceleration inversely proportional to its mass.

The Rotational Second Law in words:

The Rotational Second Law as a vector equation:

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**ROTATIONAL MOMENTUM CONSERVATION**

**13.7. FAST VS. SLOW ACTION**

Based on the activities in the previous section, you should have concluded that torque equals the time rate of change of the rotational momentum. This statement can be replaced by \( \tau = d\hat{L}/dt \). Suppose that you start a wheel spinning so that its \( \hat{L} \) vector is pointing up and that you then flip the wheel so that its \( \hat{L} \) vector points down. Which requires more torque during the “flipping time”—a fast flip or a slow one? To find out, you will need the following:

- 1 bicycle or Soap Box Derby wheel (mounted on an axle)
- 1 string (to wrap around the rim of the wheel)

| Recommended Group Size: | 4 | Interactive Demo OK?: | Y |

**13.7.1. Activity: Fast Flips and Slow Flips**

a. Which action do you predict will require more applied torque on a spinning wheel—a fast flip or a slow flip? Explain the reasons for your prediction.

b. Start a wheel spinning fairly rapidly. Try flipping it slowly and then as rapidly as possible. What do you observe about the required torques?
c. Did your prediction match your observations? If not, how can you explain what you observed?

13.8. ROTATIONAL MOMENTUM CONSERVATION

Now you can use the vector expression for Newton’s second law of Rotational Motion to show that, in theory, we expect rotational momentum on a system to be conserved if the net torque on that system is zero.

Fig. 13.8.

13.8.1. Activity: Rotational Momentum Conservation

Using mathematical arguments show that, in theory, when there is no net torque on an object or system of particles, rotational momentum is conserved.

13.9. FLIPPING A ROTATING WHEEL—WHAT CHANGES?

In Activities 13.5 and 13.7 you should have discovered that it takes a healthy torque to change the direction of the rotational momentum associated with a spinning bicycle wheel. Let’s observe a more complicated situation involving a similar change of rotational momentum. Consider a person sitting on a platform that is free to move while holding a spinning bicycle wheel. What happens if the person applies a torque to the bicycle wheel and flips the axis of the wheel by 180°? This state of affairs is shown in the following diagram.
For this observation you will need the following equipment:

- 1 platform, rotating
- 1 bicycle wheel (mounted on an axle)
- 1 piece of string (optional) (to wrap around the rim of the wheel to start it spinning)

| Recommended Group Size: | 4 | Interactive Demo OK?: | Y |

**13.9.1. Activity: What Happens When the Wheel Is Flipped?**

**a.** What do you predict will happen if a stationary person, sitting on a platform that is free to rotate, flips a spinning bicycle wheel over? Why?

**b.** What actually happens? Does the result agree with your prediction?

**c.** Use the Law of Conservation of Rotational Momentum to explain your observation in words. **Hints:** Remember that rotational momentum is a vector quantity. Does the rotational momentum of the wheel change as it is flipped? If so, how does the rotational momentum of the person and stool have to change to compensate for this?
13.10. CHANGING YOUR ROTATIONAL INERTIA

In this activity you will verify the Law of Conservation of Rotational Momentum qualitatively by rotating on a reasonably frictionless platform with your arms extended. You can then reduce your rotational inertia by pulling in your arms. This should cause you to rotate at a different rate. This phenomenon is popularly known as the ice skater effect. Since people can reconfigure themselves, they are not really rigid bodies. However, in this observation we will assume that you can behave temporarily like two rigid bodies—one with your arms extended with masses and the other with your arms pulled in with the masses.

You can observe this effect qualitatively by using the following apparatus:

- 1 rotating platform
- 2 masses, 2 kg

Recommended Group Size: 4
Interactive Demo OK?: Y
13.10.1. Activity: The Effect of Reducing Rotational Inertia

a. According to the Law of Conservation of Rotational Momentum, what will happen to the rotational speed of a person on a platform if his or her rotational inertia is decreased? Back up your prediction with equations.

b. Try spinning on the rotating platform. What happens to your rotational speed as you pull your arms in?

If you were asked to verify the Law of Conservation of Rotational Momentum quantitatively, you would need to calculate your approximate rotational inertia for two configurations. This process is a real tour de force, but it does serve as an excellent review of techniques for calculating the rotational inertia of an extended set of objects.

Ask your instructor for data on the platform’s rotational inertia. (A typical value is \( I_p = 1.0 \text{ kg m}^2 \).) Assume that each of your arms with the attached hand has a mass that is equal to a fixed percent of your total mass as shown in the following table. Idealize yourself as a cylinder (rather than a square) with long thin rods as arms. You may have to look up some data in your textbook to do the rotational inertia calculations.
Table 13.1. Percentage of the mass of a typical person’s arm and hand relative to that person’s total body mass.

<table>
<thead>
<tr>
<th></th>
<th>Single Arm/hand</th>
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<tbody>
<tr>
<td>Women</td>
<td>4.8%</td>
</tr>
<tr>
<td>Men</td>
<td>5.8%</td>
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</tbody>
</table>


13.10.2. Activity: Your Rotational Inertia

a. Find the total rotational inertia of the rotating system consisting of you, a pair of masses, and a rotating platform. Assume that you can hold the 2.0-kg masses at a distance of 5.0 cm from the axis of rotation when your elbows are in. **Hint:** Don’t forget to account for the mass and rotational inertia of the platform. Show all your work carefully.

b. Find the total rotational inertia of the rotating system if you are holding a 2.0-kg mass in each hand at arm’s length from your axis of rotation.

c. Which part of the system has the largest rotational inertia when your arms are extended, the trunk, arms, 2.0-kg masses, or the platform? Is the result surprising? Explain.